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Multiobjective Cooperative Games with Restrictions on Coalitions

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1 Introduction

Theory of cooperative games is quite useful in analyzing decision making situations along with multiple decision makers who can form coalitions. In an ordinary cooperative game (transferable utility game), the results of coalitions are described by worths of coalitions, which are real numbers. On the contrary, in a multiobjective cooperative game, the worth of each coalition is measured by multiple criteria, and therefore it is given as a set in a multidimensional real space [4, 2]. This set might be obtained by (Pareto) maximization of an admissible set [7]. Some researchers have studied multiobjective cooperative games and discussed solutions, for example the cores, of them [7, 4, 2].

On the other hand, cooperative games with some restrictions on coalitions have been actively studied recently (for example Bilbao [1] and Slikker and van den Nouweland [6]). In those cases, the set of feasible coalitions is given as a subset of the power set of the whole player set, and a new game called a restricted game which reflects the restrictions on caolitions is defined. Solutions of the original game under the restrictions on coalitions are obtained as solutions, such as the core or the Shapley value, of the restricted game.

In this paper, we consider a multiobjective cooperative game with restrictions on coalitions. We define the restricted game of the original game and discuss its properties, namely inheritance of superadditivity and convexity from the original game to the restricted game. We also study the core of the restricted game.

2 Maximum and minimum of a set in $\mathbf{R}^p$

In multiobjective optimization we consider sets in the p dimensional objective real space and maxima and/or minima of those sets. In this paper we use the following notations. First we distinguish two symbols of set inclusions: $A \subseteq B$ means that A is a subset of B, and $A \subset B$ implies that A is a proper subset of B. Let $\mathbf{R}^p$ be the p dimensional real space and $\mathbf{R}^p_+$ the nonnegative orthant in $\mathbf{R}^p$, i.e.,

$$\mathbf{R}^p_+ = \{x = (x_1, \ldots, x_p) \in \mathbf{R}^p \mid x_i \geq 0,\ i = 1, \ldots, p\}.$$

We define the sets Y_+, Y_{++}, Y_-, and Y_{--} for a set $Y \subseteq \mathbf{R}^p$ as follows:

$$\begin{aligned} Y_+ &= Y + \mathbf{R}^p_+, \quad Y_{++} = Y + (\mathbf{R}^p_+ \setminus \{\mathbf{0}\}), \\ Y_- &= Y - \mathbf{R}^p_+, \quad Y_{--} = Y - (\mathbf{R}^p_+ \setminus \{\mathbf{0}\}), \end{aligned}$$

where $\mathbf{0} = (0, \ldots, 0) \in \mathbf{R}^p$. In terms of these notations, we can define the minimum and maximum of a set in $\mathbf{R}^p$ as follows.

Definition 1 *For a set $Y \subseteq \mathbf{R}^p$, the minimum and maximum of Y are defined by*

$$\begin{aligned} \text{Min } Y &= \{y \in Y \mid (Y-y) \cap (-\mathbf{R}^p_+) = \{0\}\} = Y \setminus Y_{++} \\ \text{Max } Y &= \{y \in Y \mid (Y-y) \cap \mathbf{R}^p_+ = \{0\}\} = Y \setminus Y_{--}, \end{aligned}$$

respectively.

Remark 1 *If Y is compact, then $Y \subseteq$ [Max Y]$_-$ and hence $Y_- =$ [Max Y]$_-$ (Sawaragi et al. [5]).*

A particular type of sets in $\mathbf{R}^p$ satisfies the condition that the minimum or the maximum of a set coincides with itself.

Definition 2 *A set $Y \subseteq \mathbf{R}^p$ is said to be thin (with respect to $\mathbf{R}^p_+$) if one of the following equivalent conditions is satisfied:*

$$\begin{array}{ll} 1) & Y = \text{Min } Y \\ 2) & Y = \text{Max } Y \\ 3) & Y_+ \setminus Y = Y_{++} \\ 4) & Y_- \setminus Y = Y_{--} \end{array}$$

Equivalence of the above four conditions was proved in Tanino et al. [7].

3 Multiobjective cooperative games

An ordinary cooperative game (transferable utility game) is a pair of a set of players $N = \{1, \dots, n\}$ and a characteristic function $v : 2^N \to \mathbf{R}$ satisfying $v(\emptyset) = 0$. A subset $S \subseteq N$ is called a coalition and $v(S)$ is the worth of S. In a multiobjective cooperative game this worth should be measured by multiple (say, p throughout this paper) criteria, and therefore it is specified by a subset of $\mathbf{R}^p$ [7, 4, 2]. Thus a multiobjective cooperative game (MO-game for short) is a pair (N, V), where V is a set-valued mapping from 2^N to $\mathbf{R}^p$, i.e. $V(S) \subseteq \mathbf{R}^p$ for any $S \subseteq N$. We assume that $V(\emptyset) = \{0\}$ and that $V(S)$ is nonempty, compact and thin for any $S \subseteq N$ throughout this paper. The last condition implies that the multidimensional worth $V(S)$ of S is Pareto efficient in the MO-game. Namely there is no Pareto ordering between two points in $V(S)$. If y is contained in $V(S)_-$, then it should not be contained in $V(S)$.

We can exetnd fundamental properties of cooperative games to MO-games in straightforward and intuitive manners as follows.

Definition 3 *An MO-game (N, V) is said to be superadditive if*

$$V(S) + V(T) \subseteq V(S \cup T)_-$$

for all $S, T \subseteq N$ such that $S \cap T = \emptyset$.

Remark 2 *From the above definition, if an MO-game (N, V) is superadditive, then for any $S_k \subseteq N$ $(k \in K)$ such that $S_k \cap S_{k'} = \emptyset$ for $k \neq k'$, $\sum_{k \in K} V(S_k) \subseteq V(\bigcup_{k \in K} S_k)_-$.*

Definition 4 *An MO-game (N, V) is said to be convex if*

$$V(S) + V(T) \subseteq [V(S \cup T) + V(S \cap T)]_-$$

for all $S, T \subseteq N$.

It is obvious that convexity is a stronger requirement than superadditiveity.

4 Restricted multiobjective cooperative games by partition systems

In fundamental cooperative games and also in MO-games, it is assumed that an arbitrary subset S of N can form a coalition, i.e., every S is feasible or admissible. In practical situations, however, this assumption is not necessarily valid. Some coalitions may not be feasible because of physical or ideological reasons. Those situations are dealt with by introducing the concept of feasible coalition system [1]. A set system is a pair $(N, \mathcal{F})$, with $\mathcal{F} \subseteq 2^N$. The sets belonging to $\mathcal{F}$ are called feasible coalitions. For any $S \subseteq N$, maximal feasible subsets of S are called components of S. In many cases we impose appropriate combinatorial structures on $(N, \mathcal{F})$.

Definition 5 *([1]) A partition system is a set system satisfying*

1. *$\emptyset \in \mathcal{F}$, and $\{i\} \in \mathcal{F}$ for every $i \in N$, and*
2. *for all $S \subseteq N$, the components of S, denoted by $\Pi_{\mathcal{F}}(S) = \{T_1, \ldots T_l\}$ form a partition of S.*

Proposition 1 *([1]) A set system $(N, \mathcal{F})$ which satisfies the first condition of the above definition is a partition system if and only if $S, T \in \mathcal{F}$ and $S \cap T \neq \emptyset$ imply $S \cup T \in \mathcal{F}$.*

A typical example of partition systems is the communication structure by Myerson [3], which was discussed in detail in Slikker and van den Nouweland [6].

Definition 6 *Let (N, V) be an MO-game and let $(N, \mathcal{F})$ be a partition system. The $\mathcal{F}$-restricted game $(N, V^{\mathcal{F}})$, is defined by*

$$V^{\mathcal{F}}(S) = \mathrm{Max} \sum_{T \in \Pi_{\mathcal{F}}(S)} V(T),$$

where $\Pi_{\mathcal{F}}(S)$ is the collection of the components of $S \subseteq N$.

Remark 3 *Since $V(T)$ is compact for any $T \subseteq N$, $\sum_{T \in \Pi_{\mathcal{F}}(S)} V(T)$ is also compact. However, it is not thin generally and therefore we consider its maximum to define the restricted game. Thus $V^{\mathcal{F}}(S)$ is also compact and thin. If $S \in \mathcal{F}$, then $\Pi_{\mathcal{F}}(S) = \{S\}$ and hence $V^{\mathcal{F}}(S) = V(S)$.*

Lemma 1 *Let $(N, \mathcal{F})$ be a partition system, $S, T \subseteq N$ with $S \cap T = \emptyset$, $\Pi_{\mathcal{F}}(S) = \{S_k\}_{k \in K}$, $\Pi_{\mathcal{F}}(T) = \{T_l\}_{l \in L}$, and $\Pi_{\mathcal{F}}(S \cup T) = \{U_m\}_{m \in M}$. Then $\{S_k\}_{k \in K} \cup \{T_l\}_{l \in L}$ is a subpartition of $\{U_m\}_{m \in M}$.*

(Proof) It is obvious that $\{S_k\}_{k\in K} \cup \{T_l\}_{l\in L}$ is a partition of $\bigcup_{m\in M} U_m = S \cup T$. For each S_k there exists some U_m such that $S_k \cap U_m \neq \emptyset$. Then we can prove that $S_k \subseteq U_m$. In fact, otherwise, $U_m \subset S_k \cup U_m \in \mathcal{F}$ since $\mathcal{F}$ is a partition system, which contradicts the fact that U_m is a component of $S \cup T$. Analogously each T_l is contained in a unique U_m. Hence $\{S_k\}_{k\in K} \cup \{T_l\}_{l\in L}$ is a subpartition of $\{U_m\}_{m\in M}$. □

Due to this lemma we can prove the following theorem which shows the inheritance of superadditivity of the original game to the $\mathcal{F}$-restricted game.

Theorem 1 *Let (N, V) be a superadditive MO-game and $(N, \mathcal{F})$ be a partition system. Then the $\mathcal{F}$-restricted game $(N, V^{\mathcal{F}})$ is also superadditive.*

(Proof) Let $\Pi_{\mathcal{F}}(S) = \{S_k\}_{k\in K}$, $\Pi_{\mathcal{F}}(T) = \{T_l\}_{l\in L}$, and $\Pi_{\mathcal{F}}(S \cup T) = \{U_m\}_{m\in M}$. Then due to Lemma 1 and Remark 2,

$$\begin{aligned} V^{\mathcal{F}}(S) + V^{\mathcal{F}}(T) &= \operatorname{Max} \sum_{k\in K} V(S_k) + \operatorname{Max} \sum_{l\in L} V(T_l) \\ &\subseteq \sum_{k\in K} V(S_k) + \sum_{l\in L} V(T_l) \\ &\subseteq \sum_{m\in M} V(U_m)_- \\ &= [\sum_{m\in M} V(U_m)]_- \\ &= [\operatorname{Max} \sum_{m\in M} V(U_m)]_- \\ &= V^{\mathcal{F}}(S \cup T)_- \end{aligned}$$

The second last equality follows since $\sum_{m\in M} V(U_m)$ is compact. This completes the proof.

□

5 Inheritance of convexity

In this section we consider a more special type of feasible coalition systems called intersecting systems, and prove the inheritance of convexity to the restricted games by intersecting systems.

Definition 7 *A partition system $(N, \mathcal{F})$ is called an intersecting system if for all $S, T \in \mathcal{F}$ with $S \cap T \neq \emptyset$ we have $S \cap T \in \mathcal{F}$.*

Remark 4 *In Bilbao [1], a set system $(N, \mathcal{F})$ is called an intersecting family if for all $S, T \in \mathcal{F}$ with $S \cap T \neq \emptyset$ we have $S \cap T \in \mathcal{F}$ and $S \cup T \in \mathcal{F}$. Therefore an intersecting system is an intersecting family satisfying the first condition, $\emptyset \in \mathcal{F}$ and $\{i\} \in \mathcal{F}$, of the partition system.*

Theorem 2 *Let (N, V) be a convex MO-game and $(N, \mathcal{F})$ be an intersecting system. Then the restricted game $(N, V^{\mathcal{F}})$ is also convex.*

(Proof) Let $S, T \subseteq N$. If $S \cap T = \emptyset$, convexity reduces to superadditiveity and therefore holds obviously. Hence we assume that $S \cap T \neq \emptyset$ in the proof. Let

$$\Pi_{\mathcal{F}}(S) = \{S_1, \ldots, S_l\}, \text{ and } \Pi_{\mathcal{F}}(T) = \{T_1, \ldots, T_m\}.$$

We prove the theorem by induction both in l and in m.

a) First let $l = 1$, i.e., suppose that $\Pi_{\mathcal{F}}(S) = \{S\}$ and therefore $V^{\mathcal{F}}(S) = V(S)$. We prove the relation

$$V(S) + V(T) \subseteq [V(S \cup T) + V(S \cap T)]_-$$

by induction with respect to m. Thus we first consider the case $m = 1$. Then $\Pi_{\mathcal{F}}(T) = \{T\}$ and $V^{\mathcal{F}}(T) = V(T)$. Since $S, T \in \mathcal{F}$ and $(N, \mathcal{F})$ is an intersecting system, $S \cup T, S \cap T \in \mathcal{F}$. Then

$$\begin{aligned} V^{\mathcal{F}}(S) + V^{\mathcal{F}}(T) &= V(S) + V(T) \\ &\subseteq [V(S \cup T) + V(S \cap T)]_- \\ &= [V^{\mathcal{F}}(S \cup T) + V^{\mathcal{F}}(S \cap T)]_-, \end{aligned}$$

since (N, V) is convex. Thus we have proved the case $m = 1$. Next suppose that the result holds for $m = 1, \ldots, k-1$ $(l = 1)$ and prove the case $m = k$. Let $\Pi_{\mathcal{F}}(T) = \{T_1, \ldots, T_k\}$ and we assume without loss of generality that $S \cap T_k \neq \emptyset$. Thus $S \cup T_k, S \cap T_k \in \mathcal{F}$. Let $T' = T_1 \cup \ldots \cup T_{k-1}$. Then $\Pi_{\mathcal{F}}(T') = \{T_1, \ldots, T_{k-1}\}$.

$$\begin{aligned} V^{\mathcal{F}}(S) + V^{\mathcal{F}}(T) &= V^{\mathcal{F}}(S) + \mathrm{Max} \sum_{m=1}^{k} V(T_m) \\ &\subseteq V^{\mathcal{F}}(S) + \sum_{m=1}^{k} V(T_m) \\ &= V^{\mathcal{F}}(S) + V^{\mathcal{F}}(T_k) + \sum_{m=1}^{k-1} V(T_m) \\ &\subseteq V^{\mathcal{F}}(S \cup T_k) + V^{\mathcal{F}}(S \cap T_k) + V^{\mathcal{F}}(T') - \mathbf{R}^p_+ \\ &\subseteq V^{\mathcal{F}}(S \cup T_k \cup T') + V^{\mathcal{F}}((S \cup T_k) \cap T') + V^{\mathcal{F}}(S \cap T_k) - \mathbf{R}^p_+ \\ &= V^{\mathcal{F}}(S \cup T) + V^{\mathcal{F}}(S \cap T') + V^{\mathcal{F}}(S \cap T_k) - \mathbf{R}^p_+ \\ &\subseteq V^{\mathcal{F}}(S \cup T) + V^{\mathcal{F}}((S \cap T') \cup (S \cap T_k)) - \mathbf{R}^p_+ \\ &= [V^{\mathcal{F}}(S \cup T) + V^{\mathcal{F}}(S \cap T)]_-. \end{aligned}$$

Thus the theorem is proved for $l = 1$ and $m = 1, 2, \ldots$.

b) Now we soppose that the result is valid for $l = 1, \ldots, k-1$ and $m = 1, 2, \ldots$ and prove the case $l = k$ and m is arbitrary. In this case $\Pi_{\mathcal{F}}(S) = \{S_1, \ldots, S_k\}$. We assume without loss of generality that $S_k \cap T \neq \emptyset$ and let$S' = S_1 \cup \ldots \cup S_{k-1}$. Then

$$\begin{aligned}
V^{\mathcal{F}}(S) + V^{\mathcal{F}}(T) &= \operatorname{Max} \sum_{l=1}^{k} V(S_k) + V^{\mathcal{F}}(T) \\
&\subseteq \sum_{l=1}^{k} V(S_k) + V^{\mathcal{F}}(T) \\
&= \sum_{l=1}^{k-1} V(S_l) + V(S_k) + V^{\mathcal{F}}(T) \\
&\subseteq V^{\mathcal{F}}(S') + V^{\mathcal{F}}(S_k) + V^{\mathcal{F}}(T) - \mathbf{R}^p_+ \\
&\subseteq V^{\mathcal{F}}(S') + V^{\mathcal{F}}(S_k \cup T) + V^{\mathcal{F}}(S_k \cap T) - \mathbf{R}^p_+ \\
&\subseteq V^{\mathcal{F}}(S' \cup S_k \cup T) + V^{\mathcal{F}}(S' \cap (S_k \cup T)) + V^{\mathcal{F}}(S_k \cap T) - \mathbf{R}^p_+ \\
&= V^{\mathcal{F}}(S \cup T) + V^{\mathcal{F}}(S' \cap T) + V^{\mathcal{F}}(S_k \cap T) - \mathbf{R}^p_+ \\
&\subseteq V^{\mathcal{F}}(S \cup T) + V^{\mathcal{F}}((S' \cap T) \cup (S_k \cap T)) - \mathbf{R}^p_+ \\
&= [V^{\mathcal{F}}(S \cup T) + V^{\mathcal{F}}(S \cap T)]_-.
\end{aligned}$$

This completes the proof of the theorem. □

6 The core of restricted games

In a cooperative game, allocation scheme of the profit among the players is regarded as a solution of the game. For an MO-game, this allocation is described by np dimensional vector $x = (x^1, \ldots, x^n)$, where each x^i $(i = 1, \ldots, n)$ is a p dimensional vector representing a payoff vector received by player i.

Core is a fundamental solution concept not only in cooperative games, but also in MO-games [7, 4, 2]. It is characterized by two types of requirements: group rationality and coalition rationality.

Definition 8 *The core of an MO-game (N, V) is defined by*

$$C(V) = \{x \in \mathbf{R}^{np} \mid \sum_{i \in N} x^i \in V(N),\ \sum_{i \in S} x^i \in V(S)_+ \text{ for all } S \subset N\}.$$

Theorem 3 *Let (N, V) be an MO-game and let $(N, \mathcal{F})$ be a partition system such that $V(N)= V^{\mathcal{F}}(N)$, which is true when $N \in \mathcal{F}$. Then*

$$C(V^{\mathcal{F}}) \subseteq \{x \in \mathbf{R}^{np} \mid \sum_{i \in N} x^i \in V(N), \sum_{i \in S} x^i \in V(S)_+ \text{ for all } S \in \mathcal{F}\}$$

Moreover, if $\sum_{T \in \Pi_{\mathcal{F}}(S)} V(T)$ is thin for any $S \subseteq N$, then the equality hods in the above relation, and therefore $C(V) \subseteq C(V^{\mathcal{F}})$.

(Proof) First let $x \in C(V^{\mathcal{F}})$. Then $\sum_{i \in N} x^i \in V^{\mathcal{F}}(N) = V(N)$ and $\sum_{i \in S} x^i \in V^{\mathcal{F}}(S)_+ = V(S)_+$ for any $S \in \mathcal{F}$. Conversly, assume the thinness condition and suppose that $\sum_{i \in N} x^i \in$

$V(N) = V^{\mathcal{F}}(N)$ and $\sum_{i \in S} x^i \in V(S)_+ = V^{\mathcal{F}}(S)_+$ for all $S \in \mathcal{F}$. Take $S \notin \mathcal{F}$ and let $\Pi_{\mathcal{F}}(S) = \{S_k\}_{k \in K}$. Then

$$\sum_{i \in S} x^i = \sum_{k \in K} \sum_{i \in S_k} x^i \in \sum_{k \in K} V(S_k)_+ = [\sum_{k \in K} V(S_k)]_+ = [\mathrm{Max} \sum_{k \in K} V(S_k)]_+ = V^{\mathcal{F}}(S)_+.$$

Hence $x \in C(V^{\mathcal{F}})$, as was to be proved. □

7 Conclusion

We have defined the $\mathcal{F}$-restricted game $(N, V^{\mathcal{F}})$ for a multiobjective cooperative game (N, V) and a partition system $(N, \mathcal{F})$. It is shown that superadditivity is inherited from (N, V) to $(N, V^{\mathcal{F}})$. Inheritance of convexity is guaranteed when $(N, \mathcal{F})$ is an intersecting system. We have also considered the core of $(N, V^{\mathcal{F}})$ and proved that it can be specified by the original game (N, V) under some condition.

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